

EUCLIDEAN DE RHAM FACTOR OF A LATTICE OF NONPOSITIVE CURVATURE

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Introduction

The rigidity theorem of Mostow [18] states that if M and M^* are compact locally symmetric manifolds of nonpositive sectional curvature with no Euclidean or 2-dimensional local de Rham factors, then any isomorphism between the fundamental groups of M and M^* is induced by a diffeomorphism of M onto M^* which becomes an isometry if the metric of M or M^* is multiplied by a suitable positive constant. This result is a striking example of the following general problem: given a compact connected Riemannian manifold M of nonpositive sectional curvature, what geometric properties of M are shared by all compact connected manifolds M^* of nonpositive sectional curvature whose fundamental groups are isomorphic to that of M ?

The question above seems more reasonable if one recalls that a complete manifold M of nonpositive sectional curvature is covered topologically by a Euclidean space, and hence all homotopy groups of M except for the fundamental group are zero. In fact any isomorphism between the fundamental groups of two compact connected Riemannian manifolds M, M^* of nonpositive sectional curvature is induced by a homotopy equivalence of M to M^* . One may rephrase the question above in terms of homotopy equivalence classes of compact connected manifolds of nonpositive sectional curvature.

In previous papers [8], [11] we have given various examples of geometric properties preserved by homotopy equivalences. In some cases these geometric properties could be restated in terms of algebraic conditions on the fundamental group while in other cases no such restatement seemed possible and one apparently had to use the existence of a pseudoisometric homotopy equivalence between the two compact nonpositively curved manifolds. In [11] we obtained an extension of the Mostow rigidity theorem (cf. Corollary 4 below) which was obtained independently by Gromov [15] in even greater generality.

We also showed in [11] that the number and dimensions of the local de Rham factors are the same for homotopically equivalent compact manifolds of nonpositive sectional curvature which possess no Euclidean local de Rham factor (cf. Corollary 3 below). It was necessary in [11] to exclude the existence of Euclidean local de Rham factors since we could not prove that the dimension of the Euclidean local de Rham factor is the same for homotopically equivalent compact manifolds of nonpositive sectional curvature.

In this paper we show that the dimension of the Euclidean local de Rham factor is indeed a homotopy invariant in the class of compact manifolds of nonpositive sectional curvature. This result and others which sharpen the results described in the previous paragraph are corollaries of the following main theorem.

Theorem. *Let M be a complete Riemannian manifold of finite volume and nonpositive sectional curvature. Then the dimension of the Euclidean local de Rham factor of M equals the rank of the unique maximal abelian normal subgroup of the fundamental group of M .*

We discuss briefly the objects which appear in the statement of the theorem. If H denotes the universal Riemannian covering manifold of M , then we may write H as a Riemannian product $H_0 \times H_1 \times \cdots \times H_k$, where H_0 is a Euclidean space, possibly of dimension zero, and each H_i is irreducible for $1 \leq i \leq k$; that is, H_i cannot be written as a Riemannian product of two manifolds of positive dimension. The de Rham decomposition theorem states that this (de Rham) decomposition is unique up to the order and isometric equivalence of the factors H_i . The factors H_i are the de Rham factors of H , and the factor H_0 is the Euclidean de Rham factor of H . If M is a quotient manifold of H , then the isometric de Rham splitting occurs only locally, and the factors H_i are the local de Rham factors of M .

The unique maximal abelian normal subgroup of the fundamental group of M can be described geometrically. Regard the fundamental group of M as a discrete group Γ of isometries of H such that M is isometric to the quotient space H/Γ . The Clifford subgroup of Γ is defined to be the subgroup of Γ which consists of Clifford translations of H , those isometries ϕ of H such that the displacement function $d_\phi: p \rightarrow d(p, \phi p)$ is a constant function in H . The discussion in §1 of this paper shows that if H/Γ has finite volume, then the Clifford subgroup of Γ is the unique maximal abelian normal subgroup of Γ .

The key step in the proof of the main theorem is Lemma A of §2 whose proof is a modification of the proof of Theorem 5.1 of [20]. From the main theorem we obtain the following corollaries.

Corollary 1. *Let M, M^* be connected Riemannian manifolds of nonpositive sectional curvature and finite volume whose fundamental groups are isomorphic. Then the dimensions of the Euclidean local de Rham factors of M, M^* are equal.*

Corollary 2. *Let M be a compact connected Riemannian manifold with nonpositive sectional curvature, and suppose that the Euclidean local de Rham factor of M has positive dimension k . Then M admits a finite Riemannian covering M^{**} which is diffeomorphic to $T^k \times M_0$, where T^k denotes a k -torus, and M_0 denotes a compact manifold of nonpositive sectional curvature whose Clifford subgroup of the fundamental group is the identity.*

We remark that if M is a complete manifold of finite volume and nonpositive sectional curvature, then the center of the fundamental group of M is contained in the Clifford subgroup of the fundamental group of M . The manifold M^{**} in the corollary above need not be isometric to the Riemannian product $T^k \times M_0$. See the discussion of the main theorem of [6].

Corollary 3. *Let M, M^* denote compact connected Riemannian manifolds with nonpositive sectional curvature and isomorphic fundamental groups. Let $H = H_0 \times H_1 \times \dots \times H_k$ and $H^* = H_0^* \times H_1^* \times \dots \times H_j^*$ denote the de Rham decompositions of the universal Riemannian covering spaces of M and M^* , where H_0 and H_0^* denote the Euclidean de Rham factors. Assume moreover that the factors are ordered so that $\dim H_i \leq \dim H_{i+1}$ for $1 \leq i \leq k - 1$ and $\dim H_r^* \leq \dim H_{r+1}^*$ for $1 \leq r \leq j - 1$. Then $k = j$ and $\dim H_i = \dim H_i^*$ for $0 \leq i \leq k$.*

The result just quoted is a sharpened version of Theorem B of [11]. Briefly, one may say that the number and dimensions of the de Rham factors are homotopy invariants.

Corollary 4. *Let M, M^* denote compact connected Riemannian manifolds with nonpositive sectional curvature and isomorphic fundamental groups. Suppose that the universal Riemannian cover H^* of M^* is a reducible symmetric space of noncompact type, and that M^* is an irreducible quotient of H^* . Then M and M^* are isometric provided that one multiplies the metric of M or M^* by a suitable positive constant.*

This result is a sharpened version of Theorem A of [11]. Gromov [15] has extended the conclusion of Corollary 4 to the more general case where M^* is a compact irreducible locally symmetric manifold of rank ≥ 2 and nonpositive sectional curvature. See the introduction of [11] for further details.

We conclude by describing briefly the organization of the paper. §1 contains notation and preliminary results. §2 contains a proof of the main theorem, and §3 contains proofs of the corollaries.

1. Preliminaries

All Riemannian manifolds in this paper will be assumed to be complete connected and C^∞ and to have nonpositive sectional curvature. M will denote a nonsimply connected manifold, and H a simply connected manifold, sometimes referred to as a Hadamard manifold. All geodesics in both H and M will be assumed to have unit speed. T_1H , T_1M will denote the unit tangent bundles of H , M , and $I(H)$ the isometry group of H .

For manifolds of nonpositive sectional curvature we shall assume the notation, definitions and basic facts found in [12] and in shorter form in [5, pp. 76–78] or [10, §1].

1.1. de Rham decomposition [16]. An arbitrary Riemannian manifold N is said to be reducible if there exists a finite Riemannian covering N^* of N such that N^* is isometric to a Riemannian product $N_1^* \times N_2^*$ where each manifold N_i^* has positive dimension for $i = 1, 2$. If N is simply connected, then $N^* = N$. If N is not reducible, then it is said to be irreducible.

Let H be a reducible Hadamard manifold, and let $H_1^* \times \cdots \times H_k^*$ be any Riemannian product decomposition of H . An isometry ϕ of H is said to preserve the factors of the decomposition or to preserve the splitting if $(\phi)_* \mathfrak{N}_i(p) = \mathfrak{N}_i(\phi p)$ for each p in H and each $1 \leq i \leq k$, where \mathfrak{N}_i denotes the foliation of TH induced by the tangent spaces of H_i^* . It is not difficult to show that if ϕ preserves the splitting, then ϕ may be written uniquely as $\phi = \phi_1 \times \cdots \times \phi_k$ where $\phi_i \in I(H_i^*) \subseteq I(H)$. If $G \subseteq I(H)$ is a group of isometries such that each element of G preserves the splitting, then we define projection homomorphisms $p_i: G \rightarrow I(H_i^*)$ given by $p_i(\phi) = \phi_i$ for every $\phi \in G$ and $1 \leq i \leq k$.

It is evident that any Hadamard manifold H can be written as a Riemannian product $H_0 \times H_1 \times \cdots \times H_k$, where H_0 is a Euclidean space of dimension $r \geq 0$, and each H_i is an irreducible Hadamard manifold for $1 \leq i \leq k$. The de Rham decomposition theorem asserts that this decomposition of H is unique up to order and isometric equivalence of the factors. Such a decomposition is called the de Rham decomposition of H . The de Rham decomposition theorem is valid for an arbitrary simply connected Riemannian manifold N .

If ϕ is any isometry of H , then the differential map ϕ_* leaves invariant the Euclidean foliation of TH induced by H_0 and permutes the nonEuclidean foliations of TH which are induced by the nonEuclidean factors H_i , $1 \leq i \leq k$, in the de Rham decomposition of H . It follows that if $G \subseteq I(H)$ is any group of isometries, then there exists a subgroup \tilde{G} of finite index in G which leaves invariant all of the de Rham foliations of TH , and hence preserves the factors of the de Rham decomposition of H . By the discussion above there exist projection homomorphisms $p_i: \tilde{G} \rightarrow I(H_i)$ for $0 \leq i \leq k$.

1.2. Duality condition [2], [4], [5], [9]. For any Hadamard manifold H and any isometry group $G \subseteq I(H)$ one may define a nonwandering set $\Omega(G) \subseteq T_1H$ as follows. A vector $v \in T_1H$ is said to lie in $\Omega(G)$ if for any open set $0 \subseteq T_1H$ with $v \in 0$ we can find sequences $\{\phi_n\} \subseteq G$ and $\{t_n\} \subseteq \mathbf{R}$ such that $t_n \rightarrow +\infty$ as $n \rightarrow \infty$, and $[(\phi_n)_*T_{t_n}^{-1}(0)] \cap 0$ is nonempty for every n . Here $\{T_t\}$ denotes the geodesic flow in T_1H . The set $\Omega(G)$ is closed in T_1H and invariant under both $\{T_t\}$ and $G_* = \{\phi_* : \phi \in G\}$.

Following [2] we say that an isometry group $G \subseteq I(H)$ satisfies the duality condition in H if $\Omega(G) = T_1H$. In particular if G is a discrete group, and the quotient space H/G is a smooth manifold of finite volume, then G satisfies the duality condition. The duality condition was originally defined in the following equivalent form: a group $G \subseteq I(H)$ satisfies the duality condition if and only if for every geodesic γ in H there exists a sequence $\{\phi_n\} \subseteq G$ such that $\phi_n(p) \rightarrow \gamma(\infty)$ and $\phi_n^{-1}(p) \rightarrow \gamma(-\infty)$ as $n \rightarrow \infty$ for any point p in H . The equivalence of these two formulations of the duality condition follows from [7, Proposition 3.7].

It is proved in [9, Appendix I] that if $G \subseteq I(H)$ satisfies the duality condition, then any finite index subgroup of G also satisfies the duality condition. Moreover the duality condition is a property preserved by projection homomorphisms. To be precise let $H = H_1^* \times \cdots \times H_k^*$ be a Riemannian product manifold, and let $G \subseteq I(H)$ be a group which preserves the factors of the decomposition. Let $p_i: G \rightarrow I(H_i^*)$, $1 \leq i \leq k$, be the corresponding projection homomorphisms. Then $p_i(G)$ satisfies the duality condition in H_i^* for each $1 \leq i \leq k$ if G satisfies the duality condition in H . We remark that the groups $p_i(G)$ are in general not discrete even if the original group G is discrete.

1.3. Displacement functions [3], [10], [14]. Every isometry ϕ of a Hadamard manifold H determines a displacement function $d_\phi: H \rightarrow \mathbf{R}$ given by $d_\phi(p) = d(p, \phi p)$. The function $(d_\phi)^2$ is a C^∞ convex function on H [3, Proposition 4.2]. If d_ϕ assumes a positive minimum value on a nonempty set $A \subseteq H$, then for every point p in A [3, Proposition 4.2] also says that ϕ translates the unique geodesic of H which joins p to ϕp .

If $\Gamma \subseteq I(H)$ is a discrete group, then Γ determines a displacement function $d_\Gamma: H \rightarrow \mathbf{R}$ given by $d_\Gamma(p) = \inf_{1 \neq \phi \in \Gamma} \{d_\phi(p)\}$. The function d_Γ is continuous on H and invariant under $N(\Gamma)$, the normalizer in $I(H)$ of Γ . Since a discrete group of isometries is properly discontinuous, it follows that locally d_Γ is the minimum of finitely many displacement functions d_ϕ , $\phi \in \Gamma$. The points in H at which d_Γ is zero form a closed nowhere dense subset of H . If the quotient space H/Γ is a smooth manifold, then $d_\Gamma(p)$ is twice the injectivity radius of H/Γ at πp , where $\pi: H \rightarrow H/\Gamma$ is the projection map. For other applications of the function d_Γ see for example [10] and [14].

1.4. Clifford translations [5], [21]. An isometry ϕ of a Hadamard manifold H is called a Clifford translation if the displacement function d_ϕ is constant in H . A remark from the discussion above implies that an isometry ϕ is a Clifford translation if and only if ϕ translates the geodesic of H from p to ϕp for every point p in H . [21, Theorem 1] gives a more useful characterization of Clifford translations. Express a Hadamard manifold H as a Riemannian product $H_0 \times H_\alpha$ where H_0 is the Euclidean de Rham factor of H , and H_α is the product of all nonEuclidean de Rham factors of H . Let $p_0: I(H) \rightarrow I(H_0)$ and $p_\alpha: I(H) \rightarrow I(H_\alpha)$ be the corresponding projection homomorphisms. Then an isometry ϕ of H is a Clifford translation if and only if $p_\alpha(\phi) = 1$, and $p_0(\phi)$ is a translation of the Euclidean space H_0 . In particular H has a nontrivial Euclidean de Rham factor H_0 if $I(H)$ admits nonidentity Clifford translations.

For any group $G \subseteq I(H)$ the subgroup $C(G)$ consisting of Clifford translations in G forms an abelian normal subgroup of G . If G satisfies the duality condition in H , then $C(G)$ may be characterized as the unique maximal abelian normal subgroup of G . This latter fact may be deduced without difficulty from [13, Corollary 3] or directly from [5, Theorem 2.4].

1.5. Lattices [10], [11], [14]. A discrete group $\Gamma \subseteq I(H)$ is a uniform (nonuniform) lattice if the quotient space H/Γ is respectively compact or noncompact. A lattice Γ is reducible if the quotient space H/Γ is a reducible Riemannian manifold; that is, if there exists a finite index subgroup $\Gamma^* \subseteq \Gamma$ such that H/Γ^* is the Riemannian product of two manifolds of positive dimension. A lattice Γ is irreducible if it is not reducible.

The fact that a lattice Γ satisfies the duality condition implies by the discussion above that the Clifford subgroup $C(\Gamma)$ is the unique maximal abelian normal subgroup of Γ .

If Γ is a nonuniform lattice, then the injectivity radius of H/Γ must be arbitrarily small outside a suitably large compact subset of H/Γ . In particular for any $\varepsilon > 0$ there exists a compact subset C of H such that the displacement function d_Γ is $< \varepsilon$ in $H - \Gamma \cdot C$. See for example [10, p. 442].

2. Proof of the main theorem

See the introduction for a statement of the theorem. The first step in the proof is a result whose proof is a slight modification of the proof of [20, Theorem 5.1].

Lemma A. *Let H be an arbitrary Hadamard manifold, and express H as a Riemannian product $H_0 \times H_1$, where H_0 is the Euclidean de Rham factor of H ,*

and H_1 is the product of all nonEuclidean de Rham factors of H . Let $p_0: I(H) \rightarrow I(H_0)$ and $p_1: I(H) \rightarrow I(H_1)$ denote the induced projection homomorphisms. If Γ is a discrete subgroup of $I(H)$ that satisfies the duality condition, then $p_1(\Gamma)$ is a discrete subgroup of $I(H_1)$.

Proof. We may assume that H_0 has positive dimension, for otherwise there is nothing to prove. Let A denote the subgroup of translations of H_0 , and let G denote the closure in $I(H)$ of $\Gamma \cdot A = \{\gamma a: \gamma \in \Gamma, a \in A\}$. Note that A is a closed normal abelian subgroup of the Lie group G . By the Zassenhaus lemma as stated in [1, p. 146] it follows that G_0 , the connected component of G that contains the identity, is a solvable Lie group. One may also use [19, Theorem 8.24, p. 149] to prove that G_0 is solvable.

We assert that $p_1(G_0) = \{1\}$ or equivalently that $G_0 \subseteq I(H_0) \times \{1\} \subseteq I(H)$. Suppose that the group $p_1(G_0)$ is not the identity. Observe that $p_1(G_0)$ is solvable since G_0 is solvable, and is normalized by $p_1(\Gamma)$ since G_0 is normalized by Γ . If A^* denotes the last nonidentity subgroup in the derived series for $p_1(G_0)$, then A^* is abelian and is normalized by $p_1(\Gamma)$ since it is left invariant by all automorphisms of $p_1(G_0)$. The group $p_1(\Gamma)$ satisfies the duality condition in $I(H_1)$ by the discussion of §1 since Γ satisfies the duality condition in $I(H)$. It now follows from [5, Theorem 2.4] that $A^* \subseteq I(H_1)$ consists of Clifford translations, and hence H_1 admits a nontrivial Euclidean de Rham factor by [21, Theorem 1]. This contradicts the definition of H_1 and proves that $p_1(G_0)$ is the identity.

We show that $p_1(\Gamma)$ is discrete. Let $\phi_n = \alpha_n \times \beta_n$ be any sequence in Γ such that $\beta_n = p_1(\phi_n) \rightarrow 1$ as $n \rightarrow \infty$ and $\alpha_n = p_0(\phi_n)$. It suffices to prove that $\beta_n = 1$ for arbitrarily large values of n . Let T_n be the translation of H_0 such that $\alpha_n^* = T_n \circ \alpha_n$ fixes the origin in H_0 . If $\phi_n^* = \alpha_n^* \times \beta_n$, then $\{\phi_n^*\}$ is a bounded sequence in G and converges to an element ϕ^* in G by passing to a subsequence if necessary. If $\xi_n = \phi_{n+1}^*(\phi_n^*)^{-1}$, then $\xi_n \rightarrow 1$ as $n \rightarrow \infty$, and in particular $\xi_n \in G_0 \subseteq I(H_0) \times \{1\}$ for large n since $p_1(G_0) = \{1\}$. It follows that β_n is constant for large n since $\xi_n = \alpha_{n+1}^*(\alpha_n^*)^{-1} \times \beta_{n+1}(\beta_n)^{-1}$. Therefore $\beta_n = 1$ for large n since $\beta_n \rightarrow 1$ as $n \rightarrow \infty$. This completes the proof that $p_1(\Gamma)$ is discrete.

We shall also need the following result which strengthens slightly [10, Theorem 4.1]. The proof of the result except for occasional trivial modifications is contained in the proof of [10, Theorem 4.1].

Lemma B. *Let H be a nontrivial Riemannian product $H_1 \times H_2$, and let $\Gamma \subseteq I(H)$ be a discrete group which satisfies the duality condition and preserves the factors of the decomposition. Suppose moreover that either (a) the quotient space H/Γ is compact or (b) for every $\epsilon > 0$ there exists a compact subset*

$C \subseteq H$ such that the displacement function d_Γ is $< \varepsilon$ in $H - \Gamma \cdot C$. Finally let $\Gamma_2 = p_2(\Gamma)$ be a discrete subgroup of H_2 , where $p_1: \Gamma \rightarrow I(H_1)$ and $p_2: \Gamma \rightarrow I(H_2)$ are the projection homomorphisms. Then

(1) If $N = \text{kernel}(p_2)$, then $L(N) = H_1(\infty)$

(2) Either $\Gamma_1 = p_1(\Gamma)$ is discrete or N contains Clifford translations.

If H_1 is a Euclidean space, then N always contains Clifford translations.

See §1 for a discussion of the properties of the displacement function d_Γ . Lemma B applies to lattices [10, Theorem 4.1] since a lattice Γ satisfies property (a) if it is uniform, and property (b) if it is non-uniform. See also [10, p. 442] or [14].

We are now ready to begin the proof of the main theorem. Let M satisfy the hypotheses of the theorem, and express M as a quotient space H/Γ , where H is the universal Riemannian covering space of M , and $\Gamma \subseteq I(H)$ is the deckgroup of the covering. If M has a Euclidean local de Rham factor of dimension zero, then the subgroup of Clifford translations in Γ is the identity by [21, Theorem 1] or the discussion of §1. Therefore the only abelian normal subgroup of Γ is the identity by [5, Theorem 2.4] or the discussion of §1. The theorem is proved in this case.

It suffices to consider the case in which the Euclidean de Rham factor H_0 of H has positive dimension. Let $\bar{p}_0: \Gamma \rightarrow I(H_0)$ and $\bar{p}_1: \Gamma \rightarrow I(H_1)$ denote the projection homomorphisms where $H = H_0 \times H_1$, and H_1 is the product of all nonEuclidean de Rham factors of H . The group $\bar{p}_1(\Gamma)$ is discrete by Lemma A, and it follows from [10, Theorem 4.1] or Lemma B that Γ admits nonidentity Clifford translations.

Let $C(\Gamma)$ denote the subgroup of Γ which consists of Clifford translations, and let Γ^* denote the centralizer of $C(\Gamma)$ in Γ . By [22, Lemma 3.3] Γ^* has finite index in Γ , and hence Γ^* is also a lattice. Clearly $C(\Gamma)$ is contained in the center of Γ^* , and in fact $C(\Gamma)$ is the center of Γ^* by the discussion of §1.

Next we construct a splitting of H that decomposes H into a Riemannian product $H_1^* \times H_2^*$ such that H_1^* is a Euclidean space, $C(\Gamma) \subseteq I(H_1^*) \times \{I\}$, and $H_1^*/C(\Gamma)$ is a torus. For each $\phi \in C(\Gamma)$ let X_ϕ denote the vector field of H such that $\exp_p(X_\phi(p)) = \phi(p)$ for every point p in H . By [21] the vector field X_ϕ is parallel in H for every $\phi \in C(\Gamma)$. Let N be the distribution in H such that $N(p) = \text{span}\{X_\phi(p) : \phi \in C(\Gamma)\}$. It follows that the distributions N and N^\perp are both parallel and involutive. If H_1^* and H_2^* are maximal integral submanifolds of N and N^\perp through a fixed point p in H , then by the de Rham decomposition theorem H is isometric to the Riemannian product $H_1^* \times H_2^*$, [16]. Moreover H_1^* is an Euclidean space since any 2-plane spanned by parallel vector fields must have zero sectional curvature. If $\phi \in C(\Gamma)$ is an arbitrary

element, then for each p in H , ϕ translates the geodesic with initial velocity $X_\phi(p)$; we use [3, Proposition 4.2] and the fact that $d_\phi: q \rightarrow d(q, \phi q)$ is a constant function in H . It follows that each element $\phi \in C(\Gamma)$ leaves invariant each leaf of N . Therefore $C(\Gamma) \subseteq I(H_1^*) \times \{1\}$, and each element of $C(\Gamma)$ acts as a translation on the Euclidean space H_1^* . Clearly $H_1^*/C(\Gamma)$ is compact hence a torus by the definition of N . The arguments of this paragraph are very similar to those found in the proof of [5, Theorem 4.2]; see that paper for further details.

The fact that Γ^* centralizes $C(\Gamma)$ implies that $\phi_*N(p) = N(\phi p)$ for every p in H and every $\phi \in \Gamma^*$. In other words Γ^* preserves the splitting $H = H_1^* \times H_2^*$, and therefore every element ϕ of Γ^* can be written uniquely as $\phi = \phi_1 \times \phi_2$, where $\phi_i \in I(H_i^*)$ for $i = 1, 2$. Let $p_i: \Gamma^* \rightarrow I(H_i^*)$ be the corresponding projection homomorphisms for $i = 1, 2$.

We assert that (a) $\Gamma_1^* = p_1(\Gamma^*)$ consists of translations of the Euclidean space H_1^* and (b) the group $\Gamma_2^* = p_2(\Gamma^*)$ is a discrete group which contains no nonidentity Clifford translations. To prove (a) we observe that Γ^* centralizes $C(\Gamma)$, and hence $\Gamma_1^* = p_1(\Gamma^*)$ centralizes $p_1(C(\Gamma)) = C(\Gamma)$. The fact that H_1^* is spanned by the translations in $C(\Gamma)$ implies that every element of Γ_1^* must be a translation. To prove (b) we suppose that $\phi_2 = p_2(\phi)$ is a Clifford translation of H_2^* for some element $\phi = \phi_1 \times \phi_2$ in Γ^* . The element ϕ_1 is a translation in H_1^* by (a), and hence ϕ is a Clifford translation of H . Since $C(\Gamma) \subseteq I(H_1^*) \times \{1\}$, it follows that $\phi_2 = p_2(\phi) = 1$. The discreteness of Γ_2^* follows from [10, Lemma 5.1, p. 468] if we recall that every element in the center of Γ^* is a Clifford translation and hence lies in $I(H_1^*) \times \{1\}$.

We complete the proof of the theorem. Clearly the rank of the free abelian translation group $C(\Gamma)$ equals the dimension of H_1^* since $H_1^*/C(\Gamma)$ is compact. By the discussion of §1, $C(\Gamma)$ is the unique maximal abelian normal subgroup of Γ . Our result will be proved when we show that H_1^* is the Euclidean de Rham factor of H , and this will follow when we show that H_2^* has no Euclidean de Rham factor. We argue by contradiction, and suppose that H_2^* may be expressed as a Riemannian product $H_\alpha \times H_\beta$, where H_α has positive dimension and is the Euclidean de Rham factor of H_2^* . We wish to apply Lemma B to the discrete group Γ_2^* acting on H_2^* , and conclude that Γ_2^* admits nonidentity Clifford translations, contradicting property (b) of the previous paragraph. Clearly $\Gamma_2^* = p_2(\Gamma^*)$ satisfies the duality condition in $I(H_2^*)$ since Γ^* is a lattice and satisfies the duality condition in $I(H)$. Moreover $p_\beta(\Gamma_2^*)$ is discrete in $I(H_\beta)$ by Lemma A. If the quotient space H_2^*/Γ_2^* is compact, then Γ_2^* satisfies condition (a) of Lemma B, and we may apply Lemma B to obtain the desired contradiction.

Suppose now that H_2^*/Γ_2^* is noncompact. We are unable to show that Γ_2^* is a nonuniform lattice, but we are able to show that Γ_2^* satisfies condition (b) of Lemma B. This will allow us to apply Lemma B and obtain the contradiction described above. We verify condition (b). Let $\varepsilon > 0$ be given. The fact that H_2^*/Γ_2^* is noncompact implies that Γ^* is a nonuniform lattice, and hence there exists a compact set $C \subseteq H$ such that the displacement function d_{Γ^*} is $< \varepsilon$ in $H - \Gamma^* \cdot C$. To verify condition (b) it suffices to show that $d_{\Gamma_2^*} < \varepsilon$ in $H_2^* - \Gamma_2^* \cdot C_2$, where $C_2 = \pi_2(C)$, and $\pi_2: H = H_1^* \times H_2^* \rightarrow H_2^*$ is the canonical projection. Let $p_2 \in H_2^* - \Gamma_2^* \cdot C_2$ be given, and choose $p = (p_1, p_2) \in H$ so that $\pi_2(p) = p_2$. Clearly p lies in $H - \Gamma^* \cdot C$, and hence $d_{\Gamma^*}(p) < \varepsilon$ by the choice of C . Choose $\phi = \phi_1 \times \phi_2 \in \Gamma^*$ so that $d_\phi(p) = d(p, \phi p) < \varepsilon$. We recall that $d^2(p, \phi p) = d^2(p_1, \phi_1 p_1) + d^2(p_2, \phi_2 p_2) \geq d^2(p_2, \phi_2 p_2)$. Hence $d_{\Gamma_2^*}(p_2) \leq d_{\phi_2}(p_2) = d(p_2, \phi_2 p_2) \leq d(p, \phi p) < \varepsilon$. The group Γ_2^* satisfies condition (b) of Lemma B, and the proof of the theorem is complete.

3. Proofs of the corollaries

See the introduction for a statement of the corollaries. We omit the proof of Corollary 1, which is an immediate consequence of the theorem.

3.1. Proof of Corollary 2. Let H be the universal Riemannian covering of M , and express M as a quotient space H/Γ where Γ is a suitable uniform lattice in H . Let $C(\Gamma)$ denote the Clifford subgroup of $\Gamma \cong \pi_1(M)$, and let Γ^* denote the centralizer in Γ of $C(\Gamma)$. By the main theorem, $C(\Gamma)$ is a free abelian group of rank k , which is normal in Γ , and by the argument of [22, Lemma 3.3] it follows that Γ^* has finite index in Γ . Consider now the compact manifold $M^* = H/\Gamma^*$, which is a finite covering of M and whose fundamental group Γ^* has a center $C(\Gamma)$ which is free abelian of rank $k \geq 1$. It now follows from the main theorem of [6] that M^* admits a finite covering M^{**} with the properties listed earlier in the statement of Corollary 2.

3.2. Proof of Corollary 3. The Euclidean spaces H_0 and H_0^* have the same dimension $r \geq 0$ by Corollary 1. If $r = 0$, then Corollary 3 becomes [11, Theorem B], so we may suppose that $r \geq 1$. The idea now is to reduce to the case $r = 0$ by splitting off the Euclidean de Rham factors of H and H^* in some fashion. We use the main theorem of [6] to accomplish this.

We first reduce to the case where both H and H^* have at least one nonEuclidean de Rham factor. Let Γ, Γ^* denote uniform lattices in H, H^* such that M, M^* are the quotient spaces $H/\Gamma, H^*/\Gamma^*$. If one of the spaces, say H , were a Euclidean space, then by the Bieberbach theorems the lattice Γ would admit a subgroup $\tilde{\Gamma}$ of finite index in Γ which consisted entirely of translations

of H . An isomorphism carrying Γ onto Γ^* would carry $\tilde{\Gamma}$ onto a free abelian subgroup $\tilde{\Gamma}^*$ of finite index in Γ^* . It would thus follow that H^* is a Euclidean space whose dimension is the rank of the free abelian lattice $\tilde{\Gamma}^*$. See for example [13, Corollary 2] or [22, Corollary 1].

We have reduced to the case in which H and H^* both have an Euclidean de Rham factor of rank $r \geq 1$ and at least one nonEuclidean de Rham factor. If $G \subseteq I(H)$ denotes an arbitrary subgroup of $I(H)$, then let $C(G)$ and $Z(G)$ denote respectively the subgroup of Clifford translations in G and the center of G . Now consider the uniform lattice Γ which is the deckgroup of the compact manifold M described in the statement of Corollary 3. If Γ_0 denotes the centralizer in Γ of $C(\Gamma)$, then Γ_0 has finite index in Γ as we observed earlier in the proof of the main theorem. Moreover the discussion of §1 of [6, Lemma 3] shows that $Z(\Gamma_0) = C(\Gamma_0) = C(\Gamma)$, a free abelian group of rank $r \geq 1$.

Now let $H_\alpha = H_1 \times \cdots \times H_k$, the product of the nonEuclidean de Rham factors of H . The proof of the main theorem in this paper shows that the splitting $H = H_0 \times H_\alpha$ satisfies the conditions of [6, Lemma 1] relative to the lattice Γ_0 . The main theorem of [6] shows that Γ_0 admits a finite index subgroup Γ' such that if $\tilde{\Gamma}$ is any finite index subgroup of Γ' , then $H/\tilde{\Gamma}$ is diffeomorphic to a product $T^r \times \tilde{M}$, where $r = \dim H_0$, T^r denotes an r -torus, and \tilde{M} is a compact orientable manifold of nonpositive sectional curvature whose fundamental group has trivial center. Moreover the proof of the main theorem of [6] shows that $H_\alpha = H_1 \times \cdots \times H_k$ is the universal Riemannian cover of \tilde{M} . In particular if $\tilde{\Gamma}$ has finite index in Γ' , then $\tilde{\Gamma}$ is isomorphic to $\mathbf{Z}^r \times \tilde{G}$, where \tilde{G} denotes a uniform lattice in H_α whose center is trivial, and \mathbf{Z}^r denotes the standard integer lattice in $\mathbf{R}^r = H_0$. Similarly there exists a finite index subgroup $\Gamma^{*'} of the lattice Γ^* in H^* such that if $\tilde{\Gamma}^*$ is any finite index subgroup of $\Gamma^{*'}$, then $\tilde{\Gamma}^*$ is isomorphic to $\mathbf{Z}^r \times \tilde{G}^*$, where \tilde{G}^* is a uniform lattice in $H_\alpha^* = H_1^* \times \cdots \times H_k^*$ whose center is trivial.$

By hypothesis the lattices Γ and Γ^* are isomorphic. Let $\theta: \Gamma \rightarrow \Gamma^*$ be an explicit isomorphism. Let $\tilde{\Gamma}^* = \theta(\Gamma') \cap \Gamma^{*'}$ and $\tilde{\Gamma} = \theta^{-1}(\tilde{\Gamma}^*)$, where Γ' and $\Gamma^{*'}$ are the lattices defined in the previous paragraph. The groups $\tilde{\Gamma}$ and $\tilde{\Gamma}^*$ have finite index in Γ' and $\Gamma^{*'}$ respectively, and by the previous paragraph $\tilde{\Gamma}$ and $\tilde{\Gamma}^*$ are isomorphic respectively to $\mathbf{Z}^r \times \tilde{G}$ and $\mathbf{Z}^r \times \tilde{G}^*$, where \tilde{G} and \tilde{G}^* are uniform lattices with trivial centers in the spaces H_α and H_α^* . Since $\theta(\tilde{\Gamma}) = \tilde{\Gamma}^*$, it follows that there exists an isomorphism $\psi: \mathbf{Z}^r \times \tilde{G} \rightarrow \mathbf{Z}^r \times \tilde{G}^*$. The group \mathbf{Z}^r is the center of both $\mathbf{Z}^r \times \tilde{G}$ and $\mathbf{Z}^r \times \tilde{G}^*$ since both \tilde{G} and \tilde{G}^* have trivial centers. If $p: \mathbf{Z}^r \times \tilde{G}^* \rightarrow \tilde{G}^*$ is the projection on the second factor, then it is routine to show that $p \circ \psi: \tilde{G} \rightarrow \tilde{G}^*$ is an isomorphism. Since \tilde{G} and \tilde{G}^* are uniform lattices in $H_\alpha = H_1 \times \cdots \times H_k$ and $H_\alpha^* = H_1^* \times \cdots \times H_k^*$, we

may apply [11, Theorem B], to conclude that $k = j$ and $\dim H_i = \dim H_i^*$ for $1 \leq i \leq k$. This completes the proof of Corollary 3.

3.3. Proof of Corollary 4. By convention a symmetric space H^* of noncompact type possesses no Euclidean de Rham factor. By Corollary 1 the universal Riemannian cover H of M also possesses no Euclidean de Rham factor. The result now becomes [11, Theorem A].

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